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Finite Element Analysis of Poiseuille Flow in A Channel Bounded Below by A Permeable Bed

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Abstract

Finite Element Analysis of Poiseuille flow over permeable bed is presented in two cases, viz.,(i) the permeable bed is infinite and (ii) the permeable bed is finite. These two problems are done by Beavers & Joseph (1967).and Stephen Whitaker & Ochoa Tapia (1994). These are fundamental for the study of flows on porous media. An attempt was made to correlate the solutions of aforesaid authors with finite element analysis. Further the results obtained through FEM are found to be in good agreement with experimental results of Beavers & Joseph (1967).It is expected that these analyses may be useful for further complicate problems.

Introduction
Flow in channels with permeable boundaries is observed in many situations. Lubrication of articular cartilage with the synovial fluid is the best example. Other examples include porous bearings, ground water seepage flow, nuclear reactors, processing of composites and ceramics. An early study on boundary conditions for flow of a Newtonian fluid over permeable bed was performed by Beavers & Joseph (1967).They postulated that the slip velocity at the permeable interface differs from the mean filter velocity with in the permeable medium and the shear effects are transmitted in the body through a boundary layer region. They also proposed that the

slip velocity of the free fluid is proportional to the shear rate at permeable bed. Experimentation was performed by them and their postulation was validated.

Stephen Whitaker & Ochoa Tapia(1994) introduced Brinkman's correction into the slip condition and compared their solutions with the experimentation performed by Beavers & Joseph.

We felt that Finite Element Method could be applied to this classical problem. Formulation of the finite element model using linear fluid element was performed for both the slip conditions and also compared with experimental results. Finite Element Tool was applied through code written in C-language.

Finite Element Solution For Infinite Bed Case

The definition of this problem by Beavers & Jordon may be recalled here:

Governing equations

Flow in clear region

$$\mu \frac{\partial^2 u}{\partial y^2} + f = 0. \tag{2.1}$$

Flow in porous region described by Darcy's law

$$Q = \frac{k}{\mu} \frac{dp}{dx} \tag{2.2}$$

Boundary conditions:

$$\begin{aligned} &u = 0 at Y = L, \\ &u = u_B \ \& \ \frac{du}{dy} = \frac{\alpha}{\sqrt{k}} (u_B - Q) at Y = 0 \end{aligned} \tag{2.3}$$

Formulation of the problem using linear fluid element

Let $\hat{u}$ be the trial solution function and w(y) as the weighting function, the weighted residual statement would be

$$\int w(Y) \left(\mu \frac{\partial^2 \hat{u}}{\partial Y^2} + f \right) dY = 0 \tag{2.4}$$

Integrating by parts, the above equation becomes

$$\left[w(Y) \mu \frac{d\hat{u}}{dY} \right]_0^L - \left(\int_0^L \mu \frac{d\hat{u}}{dY} \frac{dw}{dY} dY \right) + \left(\int_0^L w(Y) f dY \right) = 0 \tag{2.5}$$

using the symbol 'Y' for global coordinate running along the length of the entire gap between plates and the symbol 'y' for representing the local coordinate system within each element 0 to l, the discretization leads to

$$\sum_{k=1}^{k=n}\left(\int_0^l \mu \frac{d\hat{u}}{dy}\frac{dw}{dy}dy\right) = \sum_{k=1}^{k=n}\left(\int_0^l w(y)\, f dy\right) + \sum_{k=1}^{k=n}\left[w(y)\mu \frac{du}{dy}\right]_0^l \tag{2.6}$$

Linear element is a one dimensional 2- noded element with one degree of freedom at each node. Further the element can take axial loads only. The pictorial summary of such an element is shown below:

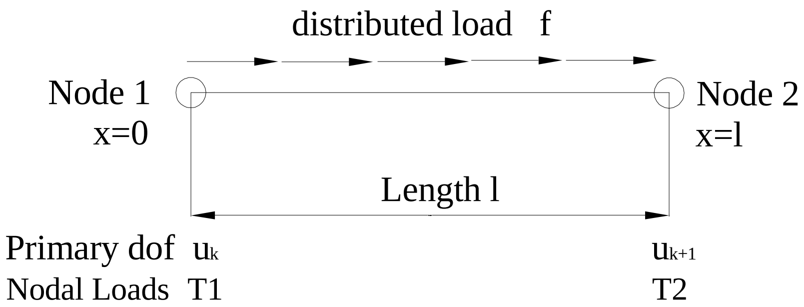

Figure 2.1: A Linear Fluid Element.

The trial velocity solution for this linear element is

$$\hat{u}(y) = (1 - y/l)u_k + (y/l)u_{k+1} \tag{2.7}$$

It is to be observed that the shape functions $N_1 = (1 - y/l)\ \&\ N_2 = (y/l)$ are both linear. For Galerkin's method, weighting functions are chosen to be same as the trial solution functions. Hence $w_1(y) = N_1 and w_2(y) = N_2$.

Making the substitutions the weighted residual statement shown by equation (9) reduces to the finite element equation:

$$\frac{\mu}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_k \\ u_{k+1} \end{bmatrix} = \frac{fl}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\tau_0{}^k \\ \tau_l{}^k \end{bmatrix} \tag{2.8}$$

The matrix on the L.H.S is the element stiffness matrix .The first term on R.H.S is the distributed load vector due to pressure gradient while the second term represents the point load vector due to shear stress

Solution Procedure
Discretisation

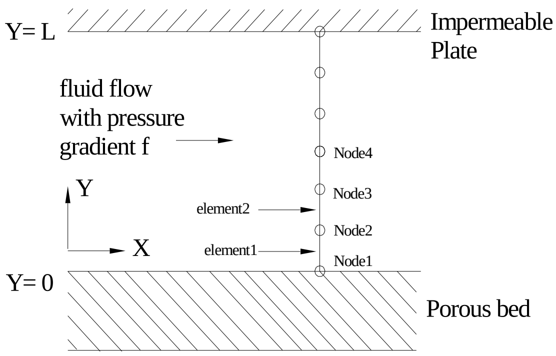

Figure 2.2: Finite Element Model.

The assembly of element equations is quite simple because the elements are in a line.

Boundary conditions are implemented through penalty approach:

$$\overline{K_{ii}} = K_{ii} + a_p \;\;\&\;\; \overline{F_i} = a_p \times \overline{u_i} \qquad (2.9)$$

$a_p = 10^3 to 10^6 \left|\max K_{ij}\right|$ is the penalty number .

Solution of the assembly of equations is performed by Gauss elimination.

Convergence of solution

The 1-d fluid element used here is checked for convergence of solution. This is shown in the following diagram.

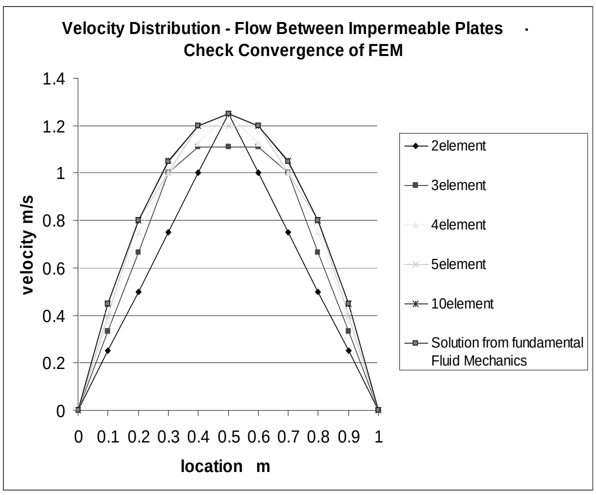

Figure 2.3

Comparison

Velocity Field

The exact solution for this problem according to Beavers & Joseph (Ref.1) is

$$u = u_B\left(1+\frac{\alpha}{\sqrt{k}}Y\right)+\frac{1}{2\mu}\left(Y^2+2\alpha Y\sqrt{k}\right)\frac{dp}{dx}$$
$$u_B = \frac{-k}{2\mu}\left(\frac{\sigma^2+2\alpha\sigma}{1+\alpha\sigma}\right)\frac{dp}{dx} \qquad (2.10)$$

Where u is the velocity component in X- direction, u_B is the slip velocity at the permeable bed and

$$\sigma = \frac{h}{\sqrt{k}} \qquad (2.11)$$

Mass Flow Rate

The mass flow rate per unit width of the channel is given by

$$M = -\frac{\rho L^3}{12\mu}\left(\frac{dp}{dx}\right)-\frac{\rho L^3}{4\mu\sigma}\left(\frac{\sigma+2\alpha}{1+\alpha\sigma}\right)\left(\frac{dp}{dx}\right) \qquad (2.12)$$

Fractional Increase In Flow Rate

The fractional increase in mass flow rate through the channel when compared with impermeable bounding top and bottom plates is

$$\phi = \frac{3(\sigma+2\alpha)}{\sigma(1+\alpha\sigma)} \qquad (2.13)$$

The comparisons have been shown below

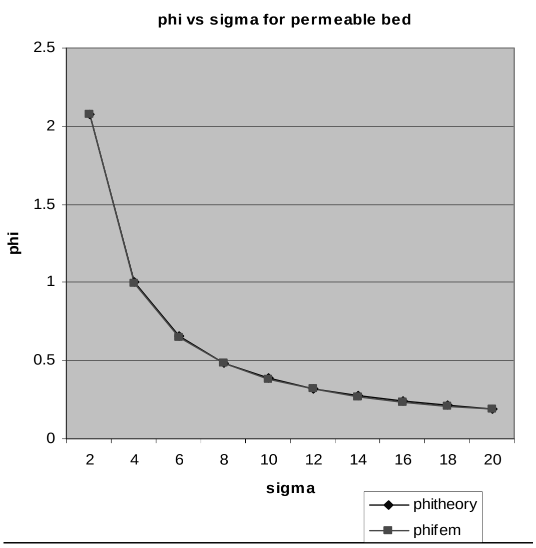

Figure 2.4

We observe that there is good agreement between FEM theory and experimental result of Beavers & Joseph(1967).

Finite Element Solution for Finite Bed Case

Governing equations

Figure (3.1) shows the clear fluid and porous regions with coordinate system starting from top plate (impermeable plate).

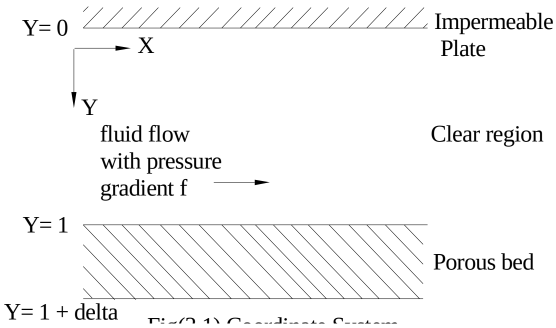

Figure 3.1: Coordinate System.

Region 1 (clear fluid region)

Governing differential equation

$$\mu \frac{\partial^2 u_1}{\partial y^2} = \frac{dp}{dx}. \tag{3.1}$$

Region 2 (porous fluid region)

Governing differential equation

$$\mu_{eff} \frac{\partial^2 u_2}{\partial y^2} - \frac{\mu}{k} u_2 = \frac{dp}{dx}. \tag{3.2}$$

Boundary conditions

At y=0, u1 = 0 (3.3)

at y=h, u1=u2 (3.4)

at y=h, $\mu_{eff}\left(\frac{du_2}{dy}\right) - \mu\left(\frac{du_1}{dy}\right) = \beta\left(\frac{\mu}{\sqrt{k}}\right)u_2$ (3.5)

at y=h + delta , u2 = 0. (3.6)

Non-dimensionalisation
Writing

$$D_a = \frac{k}{h^2}, \overline{u_i} = \frac{u_i}{u_{av}}, \mathrm{Re} = \frac{\rho u_{av} h}{\mu}, \quad (3.7)$$
$$\overline{y} = \frac{y}{h}, p = \frac{p}{\rho u_{av}^2}, \overline{x} = \frac{x}{h}, \overline{\delta} = \frac{\delta}{h}$$

The non-dimensionalised governing equations and boundary conditions are as follows:

Region 1 (clear fluid region)
Governing differential equation

$$\frac{\partial^2 u_1}{\partial y^2} = R_e \frac{dp}{dx}. \quad (3.8)$$

Region2 (porous fluid region)
Governing differential equation

$$\frac{\partial^2 u_2}{\partial y^2} - \frac{\varepsilon}{D_a} = \varepsilon R_e \frac{dp}{dx}. \quad (3.9)$$

Boundary conditions

at y=0, u1 = 0

at y=1, u1 =u2

at y=1 , $\left(\frac{du_2}{dy}\right) - \varepsilon\left(\frac{du_1}{dy}\right) = \beta\left(\frac{\varepsilon}{\sqrt{D_a}}\right)u_2$ (3.10)

at y=1+delta , u2 = 0.

Note: Non-dimensionalised representation – bar over the symbols has been removed for ease of handling .

Finite Element Formulation Using Linear Fluid Elements

Since the flow is governed by different conditions in the clear and porous regions, the formulation is done separately for each region.

For Clear region
The element equation for linear fluid element described by equation (2.11) of chapter 2 is also non-dimensionalised to give the following element equation.

$$\frac{1}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_k\\ u_{k+1}\end{bmatrix}=\frac{Gl}{2}\begin{bmatrix}1\\ 1\end{bmatrix}+\begin{bmatrix}-\frac{du}{dy}/\ for y=0\\ \frac{du}{dy}/\ for y=l\end{bmatrix} \tag{3.11}$$

Note

$$G = R_e\frac{dp}{dx} \tag{3.12}$$

For Porous Region

$$\int w(Y)\left(\frac{\partial^2 u_2}{\partial y^2}-\frac{\varepsilon}{D_a}=\varepsilon G.\right)dY\ = 0 \tag{3.13}$$

Integrating by parts, the above equation becomes

$$\left[w(Y)\frac{du_2}{dY}\right]_0^L-\left(\int_0^L\frac{du}{dY}\frac{dw}{dY}dY\right)-\frac{\varepsilon}{D_a}\int_0^L w(Y)u_2dy-\varepsilon G\left(\int_0^L w(Y)dY\right)=0 \tag{3.14}$$

using the symbol ‘Y’ for global coordinate running along the length of the entire gap between plates and the symbol ‘y’ for representing the local coordinate system within each element, non-dimensionalised, the discretization leads to

$$\sum_{k=1}^{k=n}\left(\int_0^l\frac{du_2}{dy}\frac{dw}{dy}dy\right)+\frac{\varepsilon}{D_a}\int_0^L w(y)u_2dy-=\sum_{k=1}^{k=n}\left(\int_0^l w(y)G\varepsilon dy\right)+\sum_{k=1}^{k=n}\left[w(y)\frac{du_2}{dy}\right]_0^l \tag{3.15}$$

On similar lines, the trial velocity solution for linear fluid element can be written as $u(y) = (1-y/l)u_k + (y/l)u_{k+1}$.

Note the shape functions $N_1 = (1-y/l)\,\&\,N_2 = (y/l)$ are both linear. Following Galerkin’s method, weighting functions are chosen to be same as the trial solution functions. Hence $w_1(y) = N_1 and w_2(y) = N_2$.

Making the substitutions the weighted residual statement shown by equation (4.15) reduces to the finite element equation:

$$\frac{1}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_k\\ u_{k+1}\end{bmatrix}+\frac{\varepsilon l}{6D_a}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}u_k\\ u_{k+1}\end{bmatrix}=\frac{\varepsilon Gl}{2}\begin{bmatrix}1\\ 1\end{bmatrix}+\begin{bmatrix}-\frac{du_2}{dy}/\ for y=0\\ \frac{du_2}{dy}/\ for y=l\end{bmatrix} \tag{3.16}$$

The two matrices on the L.H.S constitute the element stiffness matrix .The first term on R.H.S is the distributed load vector due to pressure gradient while the second term represents the point load vector due to shear stress.

Solution Procedure

The procedure is same as that has been performed previously. However separate discretization is done for clear and porous regions and separate code has been written for the two regions.

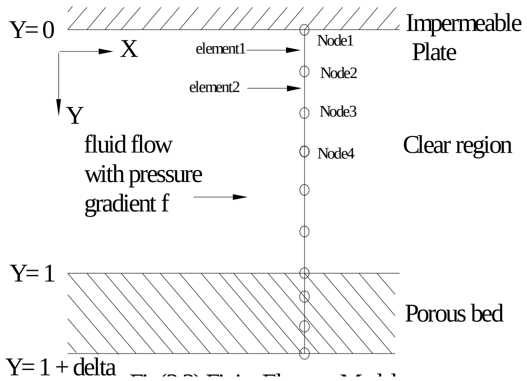

Figure 3.2: Finite Element Model.

Step2 : Write element equation for each element
Using equation (4.11), element equation for each element is written for clear region while equation (4.16) is used for porous region in a similar way.

Step3 : Assemble the element equations
Since elements are in a line, assembly becomes simple. Stiffnesses at common nodes are added. Similarly loads at common nodes are added. The assembled equation for the clear region now becomes

$$\frac{\mu}{l}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1+1 & -1 & 0 & 0 \\ 0 & -1 & 1+1 & -1 & 0 \\ 0 & 0 & -1 & 1+1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \frac{fl}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -\tau_0^1 \\ 0 \\ 0 \\ 0 \\ \tau_l^4 \end{bmatrix} \tag{3.17}$$

The assembled equation for the porous region now becomes

$$\left(\frac{1}{l}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1+1 & -1 & 0 & 0 \\ 0 & -1 & 1+1 & -1 & 0 \\ 0 & 0 & -1 & 1+1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} + \frac{\varepsilon l}{6D_a}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2+2 & 1 & 0 & 0 \\ 0 & 1 & 2+2 & 1 & 0 \\ 0 & 0 & 1 & 2+2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \frac{\varepsilon Gl}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{du_2^{(1)}}{dy} / for\ y=0 \\ 0 \\ 0 \\ 0 \\ \frac{du_2^{(4)}}{dy} / for\ y=1 \end{bmatrix} \tag{3.18}$$

(Note the above assembly of element equations is for 4 elements)

The Boundary conditions are implemented through penalty approach:

$$\overline{K_{ii}} = K_{ii} + a_p \ \ \& \ \ \overline{F_i} = a_p \times \overline{u_i} \tag{3.19}$$

$a_p = 10^3 \, to \, 10^6 \left| \max K_{ij} \right|$ is the penalty number .

Solution of the assembly of equations is performed by Gauss elimination.

Comparison

The exact solution for the governing equation according J.A. Ochoa-Tapia and S. Whitaker is

$$u_1 = \frac{GY^2}{2} + C_1 Y$$
$$u_2 = C_3 e^{lY} + C_4 e^{-lY} - \frac{G\varepsilon}{l^2} \quad (3.20)$$

The constants are given as follows

$$C_4 = \frac{G\varepsilon\left[\left(-0.5 + D_a + \beta\sqrt{D_a}\right) + \frac{1}{l^2}\left(l - D_a - \beta\sqrt{D_a}\right)e^{-l\delta}\right]}{e^{-l}\left[\left(l + \varepsilon + \frac{\beta\varepsilon}{\sqrt{D_a}}\right) + \left(l - \varepsilon - \frac{\beta\varepsilon}{\sqrt{D_a}}\right)e^{-2l\delta}\right]}$$
$$C_3 = -C_4 e^{-2l(1+\delta)} + GD_a e^{-l(1+\delta)} \quad (3.21)$$
$$C_1 = C_3 e^{l} + C_4 e^{-l} - GD_a - \frac{G}{2}$$

u_1 refers to the velocity distribution in clear region and u_2 refers to the velocity distribution in porous region.

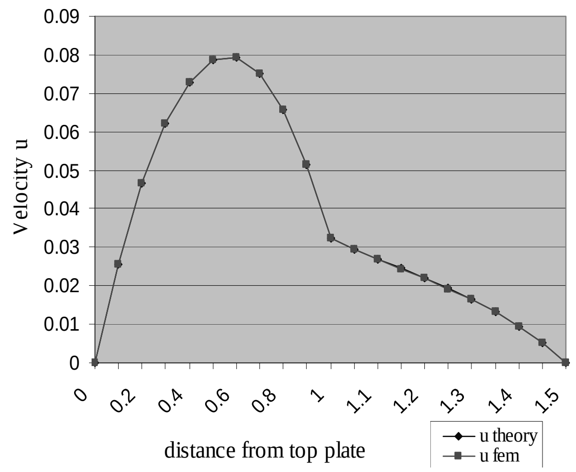

Figure 3.5: Velocity distribution in complete region.

Algorithm for the C-program we used to implement fem

1. Input number of elements, viscosity of the fluid, pressure gradient f, density of

fluid, permeability of bed k, bed structure α, gap in the channel h.
2. Initialize the stiffness matrix A[i][j] terms and velocities at nodes to zero
3. Calculate the bed velocity given by equation 2.10.
4. Calculate non-zero – principal diagonal terms (first, last & middle terms separately) and remaining carpet terms of the global stiffness matrix.
5. Compute penalty a_p=max|A[i][j]|$10^{4.}$
6. Form augmented matrix [A | b] where b is the Nodal load vector.
7. Gauss elimination – perform row operations to form upper triangular matrix, substitute bottom up to get the primary dof [u].
8. Compute the mass flow rate mfr by summing up mass flow rate via each element.
9. Calculate relative increase in mass flow rate φ.
10. The steps are repeated for different values of σ to get the plot fractional increase in flow φ with respect to σ.

Conclusions

Formulation of the finite element model using linear fluid element was found to be successful in obtaining the velocity distributions for both the slip conditions and also compared with experimental results. Better convergence is observed even with lesser number of elements.

References

[1] Gordon S. Beavers & Daniel D. Joseph, 1967,“Boundary conditions at a naturally Permeable wall” Journal of Fluid mechanics, Vol.30, part 1 pages 197 – 207.

[2] Donald F. Young & Theodore H. Okiishi, Bruce R Munson, 2002, “Fundamentals of Fluid Mechanics” John Wiley & Sons(Asia) Pvt. Ltd., 4th edition.

[3] J.Albert Ochoa-Tapia & Stephen Whitaker, 1995, “Momentum transfer at the boundary between a porous medium and a homogeneous fluid – I. Theoretical development”, International Journal of Heat and Mass Transfer, Vol:38 No.14 pp 2645 – 2646.

[4] J.Albert Ochoa-Tapia & Stephen Whitaker, 1995, “Momentum Transfer at the boundary between a porous medium and a homogeneous fluid – II. Comparison with experiment” - International Journal of Heat and Mass Transfer Vol:38 No.14 pp 2647 – 2655.

[5] Dr.P.Seshu,2004, “A Text Book of Finite Element Analysis”, Prentice Hall of India Pvt Ltd, NewDelhi First Edition

[6] Dr.J.N.Reddy,2005, “A Introduction to Finite Element Method”, Tata McGraw-Hill 3rd Edition 2005

[7] Tirupathi R. Chandrupatla & Ashok D. Belegundu, 2002, “Introduction to Finite elements in Engineering”, Pearson education ,Third edition ,

[8] Chandrakant S. Desai & John F.Abel, 1987, "Introduction to the Finite Element Method, A Numerical Method for Engineering Analysis " CBS Publishers & Distributors New Delhi, First Edition.

[9] N. Rudraiah, R. Veerabadraiah, B.C. Chandra Sekhar, S.T. Nagaraj, 1979 "Some Flow Problems In Porous Media", PGSAM Series, Vol II by, Bangalore University, Bangalore.